Spatiotemporal wave and soliton solutions to the generalized (3+1)-dimensional Gross-Pitaevskii equation

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Exact extended traveling wave and spatiotemporal soliton solutions to the generalized (3+1)-dimensional Gross-Pitaevskii equation with time-dependent coefficients are obtained. The case with constant diffraction and parabolic potential strength, but with variable gain, is discussed in some detail. It is found that gain in the system is necessary for the appearance of stable solitons.

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I. INTRODUCTION

Gross-Pitaevskii equation (GPE) is of tremendous importance in Bose-Einstein condensation (BEC), where it describes the behavior of the condensate wave function [1]. It has been introduced independently by Gross [2] and Pitaevskii [3] for an unrelated problem, but has since been found of great use in BEC. In addition, it has been used in the studies of superfluidity in liquid He II, as well as of pulse propagation in nonlinear (NL) optics [4]. Solutions to GPE are of great interest, because they can be applied to a diverse array of quantum systems. Among other, solitary wave solutions [5] have been noted in GPE. However, stable exact soliton solutions to GPE exist only in (1+1) dimensions [(1+1)D] [6,7]; there are no known *exact* stable solitons in higher dimensions. In a variational and numerical treatment, Adhikari has shown that the three-dimensional (3D) spatiotemporal (ST) optical solitons can be stabilized by a rapidly oscillating scattering length or the dispersion coefficient in a Kerr medium with cubic nonlinearity [8].

Here, we present *analytical* traveling wave and soliton solutions to the GPE in (3+1)D; that is, in three spatial dimensions and time. The solutions we find depict the way in which an initial traveling wave packet obeying GPE changes in time; such solutions are necessarily *transient* in nature. This is the consequence of not only the equation being of the time-dependent Schrödinger type, but also of the fact that the coefficients in the equation are time-dependent, which is typical of BEC. Hence, the solutions might diminish in time, or blow up, or oscillate, or converge to a specific spatial form; it is the latter two forms of solutions that we are mostly interested in.

An unrelated, but nonetheless very important aspect of the problem, is the *stability* of these solutions; that is, how do they evolve in time when *disturbed* from their analytically given forms. This aspect of the problem must be addressed numerically (with the help of the linear stability analysis) and will be presented elsewhere. It suffices to mention that the solutions found here depend on the fine balance between different terms in GPE, however their stability in (2+1)D and (3+1)D can conveniently be addressed by the dispersion and nonlinearity management methods [9]. It is expected that the stability of localized multidimensional solutions will be enhanced in GPE systems with oscillating dispersion/diffraction and/or sign-changing nonlinearity [8,10].

The paper is composed in the following manner. Section II introduces (3+1)D GPE and the method of solution. Section III presents solutions and Sec. IV brings discussion on some salient features of the solutions. Section V concludes the paper.

II. ANALYSIS

We consider GPE in (3+1)D with *distributed* coefficients [1]

$$i\partial_t u + \frac{\beta(t)}{2}\Delta u + \chi(t)|u|^2 u + \alpha(t) r^2 u = i\gamma(t) u. \tag{1}$$

Here, t is time, $\Delta=\partial_x^2+\partial_y^2+\partial_z^2$ is the 3D Laplacian, $r=\sqrt{x^2+y^2+z^2}$ is the position coordinate, and $\alpha(t)$ stands for the strength of the quadratic potential as a function of time. It is strictly assumed that $\alpha(t)\neq 0$; otherwise, we are back to the generalized NL Schrödinger equation, which has already been discussed in [10,11]. The functions β, χ, and γ stand for the diffraction, nonlinearity, and gain coefficients, respectively. All coordinates in Eq. (1) are made dimensionless by the choice of coefficients.

We define the complex field u of Eq. (1) in terms of its amplitude and phase:

$$u(x,y,z,t) = A(x,y,z,t)\exp[iB(x,y,z,t)]. \tag{2}$$

Substituting u into Eq. (1), two coupled equations for A and B are obtained

$$\partial_z A + \frac{\beta}{2}[2\partial_x A\partial_x B + 2\partial_y A\partial_y B + 2\partial_z A\partial_z B + A\Delta B] = \gamma A, \tag{3}$$

$$-A\partial_z B + \frac{\beta}{2}[\Delta A - A((\partial_x B)^2 + (\partial_y B)^2 + (\partial_z B)^2)] + \chi A^3 + \alpha r^2 A = 0. \tag{4}$$

To these equations we apply the balance principle [12,13] and the F-expansion technique [14,15], as developed in [11].

TABLE I. Jacobi elliptic functions.

	c_0	c_2	c_4	F	$M=0$	$M=1$
1	1	$-(1+M^2)$	M^2	sn	sin	tanh
2	$1-M^2$	$2M^2-1$	$-M^2$	cn	cos	sech
3	M^2-1	$2-M^2$	-1	dn	1	sech
4	M^2	$-(1+M^2)$	1	ns	cosec	coth
5	$-M^2$	$2M^2-1$	$1-M^2$	nc	sec	cosh
6	-1	$2-M^2$	M^2-1	nd	1	cosh
7	1	$2-M^2$	$1-M^2$	sc	tan	sinh
8	$1-M^2$	$2-M^2$	1	cs	cot	cosech
9	1	$-(1+M^2)$	M^2	cd	cos	1
10	M^2	$-(1+M^2)$	1	dc	sec	1

We seek the traveling wave solutions to Eqs. (3) and (4), and assume the functions to be of the form

$$A = f(t)F(\theta) + g(t)F^{-1}(\theta), \tag{5}$$

$$\theta = k(t)x + l(t)y + m(t)z + \omega(t), \tag{6}$$

$$B = a(t)r^2 + b(t)(x+y+z) + e(t), \tag{7}$$

where f, g, k, l, m, ω, a, b, and e are parameter functions to be determined, and F is a Jacobi elliptic function (JEF).

Substituting Eqs. (5)–(7) into Eqs. (3) and (4) and requiring that x^qF^n, y^qF^n, and t^qF^n, $(q=0,1,2,\ n=0,1,2,3)$ of each term be separately equal to zero, a system of algebraic or first-order ordinary differential equations for the parameter functions is determined

$$\frac{df_j}{dt} + 3a\beta f_j - \gamma f_j = 0, \tag{8}$$

$$\frac{dk}{dt} + 2ka\beta = 0, \tag{9}$$

$$\frac{dl}{dt} + 2la\beta = 0, \tag{10}$$

$$\frac{dm}{dt} + 2ma\beta = 0, \tag{11}$$

$$\frac{da}{dt} + 2\beta a^2 - \alpha = 0, \tag{12}$$

$$\frac{db}{dt} + 2\beta ab = 0, \tag{13}$$

$$\frac{d\omega}{dt} + \beta(k+l+m)b = 0, \tag{14}$$

$$\frac{de}{dt} + \frac{\beta}{2}[3b^2 - (k^2+l^2+m^2)c_2] - 3\chi f_1 f_2 = 0, \tag{15}$$

$$f_1[\beta(k^2+l^2+m^2)c_4 + \chi f_1^2] = 0, \tag{16}$$

$$f_2[\beta(k^2+l^2+m^2)c_0 + \chi f_2^2] = 0, \tag{17}$$

where $j=1,2$, $f_1=f$, and $f_2=g$. The constants c_0, c_2, and c_4 in Eqs. (15)–(17) are related to the elliptic modulus M of JEFs (see Table I). By solving Eqs. (8)–(17) self-consistently, one obtains a set of conditions on the coefficients and parameters, necessary for Eq. (1) to have exact traveling wave and ST soliton solutions [10]. Note the importance of the parameter $a(t)$, which is known as the *chirp* function. All other parameters, as well as the solutions to GPE, explicitly or implicitly depend on a.

The existence of the coefficient $\alpha(t)$ makes the solution of Eq. (12) for the chirp significantly more difficult than that of the corresponding equation in the case of the generalized NL Schrödinger equation, see [10,11]. Since the solution of $a(t)$ also determines the solutions of all other parameters, the solutions obtained here are markedly different from those in [10,11]. Indeed, Eq. (12) is of the Riccati equation type, which has no analytical solutions for the general functions $\alpha(t)$ and $\beta(t)$; nevertheless its numerical solution entails little difficulty. However, for certain choices of $\alpha(t)$ and $\beta(t)$ it is possible to obtain exact solutions. Here, we will report on the exact solutions when these two functions are constant. A more complete analysis will be presented elsewhere.

III. SOLUTION

We consider the most generic case for $\alpha(t)=\alpha$ and $\beta(t)=\beta$, where α and β are arbitrary positive real constants, f and g assumed nonzero, and $\gamma(t)$ is an arbitrary function. The following set of exact solutions is found:

$$f = f_0\left[\frac{e^{pt/2}(1+C)}{1+Ce^{pt}}\right]^{3/2}\exp\left(\int_0^t \gamma dt\right), \quad g = \epsilon\sqrt{\frac{c_0}{c_4}}f; \tag{18}$$

$$k = \frac{e^{pt/2}(1+C)}{1+Ce^{pt}}k_0, \quad l = \frac{e^{pt/2}(1+C)}{1+Ce^{pt}}l_0, \quad m = \frac{e^{pt/2}(1+C)}{1+Ce^{pt}}m_0; \tag{19}$$

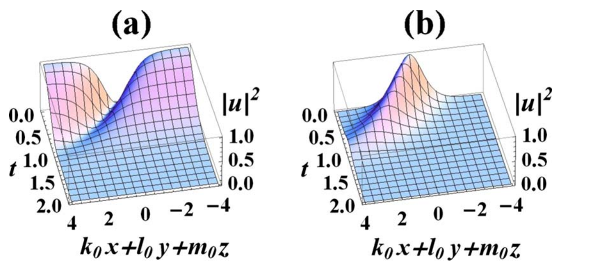

FIG. 1. (Color online) Decaying bent soliton solutions to GPE as functions of time, for $b_0=1$. Intensity $|u|^2$ for (a) $F=\tanh$ and (b) $F=\text{sech}$ presented as functions of $k_0x+l_0y+m_0z$ and t. Coefficients: $\beta=1$, $\alpha=1$, $\gamma(t)=-0.05$, $a_0=0$, $e_0=0$, $k_0=l_0=m_0=1$, $\omega_0=0$, and $\epsilon=0$.

$$\omega=\omega_0-\beta(k_0+l_0+m_0)b_0\frac{(1+C)(e^{pt}-1)}{p(1+Ce^{pt})}; \tag{20}$$

$$a=\sqrt{\frac{\alpha}{2\beta}}\frac{Ce^{pt}-1}{Ce^{pt}+1},\quad b=\frac{e^{pt/2}(1+C)}{1+Ce^{pt}}b_0; \tag{21}$$

$$e=e_0+\frac{\beta}{2}[(k_0^2+l_0^2+m_0^2)(c_2-6\epsilon\sqrt{c_0c_4})-3b_0^2]\times\frac{(1+C)(e^{pt}-1)}{p(1+Ce^{pt})}; \tag{22}$$

where $C=(\sqrt{\frac{\alpha}{2\beta}}+a_0)/(\sqrt{\frac{\alpha}{2\beta}}-a_0)$ and $p=2\sqrt{2\alpha\beta}$. The subscript 0 denotes the value of the given function at $t=0$. It is assumed that $a_0\neq\sqrt{\frac{\alpha}{2\beta}}$. When $a_0=\sqrt{\frac{\alpha}{2\beta}}$ one obtains the appropriate solution expressions by taking the limit $C\to\infty$. A parameter $\epsilon=\pm1$ is introduced in Eqs. (18) and (22), to distinguish the two present possibilities.

It should also be noted that $\chi(t)$ is not arbitrary, but depends on α, β, and $\gamma(t)$

$$\chi(t)=-\beta c_4(k_0^2+l_0^2+m_0^2)f_0^{-2}\left[\frac{e^{pt/2}(1+C)}{1+Ce^{pt}}\right]^{-1}\times\exp\left[-2\int_0^t\gamma(t)dt\right]. \tag{23}$$

This equation should be understood as an *integrability* condition on Eq. (1) for solution by the present method.

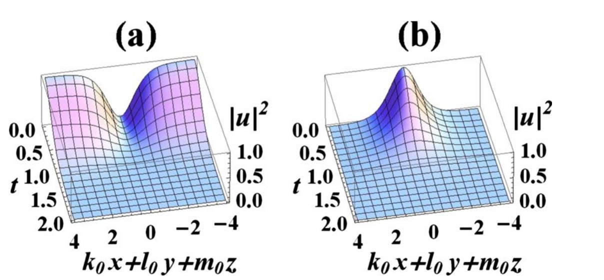

FIG. 2. (Color online) Decaying straight soliton solutions to GPE as functions of time. The setup and parameters are the same as in Fig. 1 except for $b_0=0$.

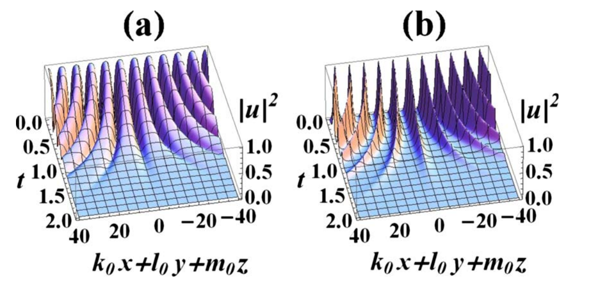

FIG. 3. (Color online) Decaying traveling wave solutions, given in terms of JEFs. The setup and parameters are the same as in Fig. 2, except for $M=0.99$. (a) $F=\text{sn}$ and (b) $F=\text{cn}$.

Incorporating Eqs. (18)–(22) back into Eqs. (5)–(7), we obtain the general periodic traveling wave and soliton solutions to GPE

$$u=f_0\left[\frac{e^{pt/2}(1+C)}{1+Ce^{pt}}\right]^{3/2}\exp\left(\int_0^t\gamma dt\right)\left[F(\theta)+\epsilon\sqrt{\frac{c_0}{c_4}}\frac{1}{F(\theta)}\right]\cdot\exp i[a(x^2+y^2+z^2)+b(x+y+z)+e], \tag{24}$$

where

$$\theta=\omega_0+kx+ly+mz-\beta(k_0+l_0+m_0)b_0\frac{(1+C)(e^{pt}-1)}{p(1+Ce^{pt})}. \tag{25}$$

Apart from the solutions given in Eqs. (18)–(22), one can alternatively assume that $g=0$, in which case one obtains the exact same equations to which Eqs. (18)–(22) would reduce for $\epsilon=0$. Thus, the parameter ϵ in Eq. (24) can assume three values: ±1 and 0.

IV. DISCUSSION

There are few key differences between the solutions obtained here and the ones obtained in [10]. Most notably, there is no meaningful distinction, in the sense of chirp *vs.* no chirp, between the solutions with $a_0\neq0$ and $a_0=0$. The value of $a_0=0$ does not entail any special status; instead, it is the value of $\sqrt{\frac{\alpha}{2\beta}}$ that is of some importance. For $a_0>-\sqrt{\frac{\alpha}{2\beta}}$, a converges to $\sqrt{\frac{\alpha}{2\beta}}$ as t increases; for $a_0=\pm\sqrt{\frac{\alpha}{2\beta}}$, it stays constant and for $a_0<-\sqrt{\frac{\alpha}{2\beta}}$ there are singularities in the param-

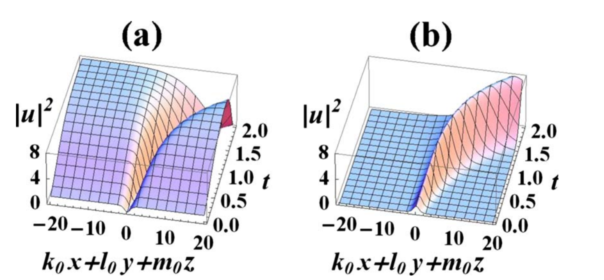

FIG. 4. (Color online) Bent soliton solutions as functions of time. (a) Dark; (b) Bright. The setup and parameters are the same as in Fig. 1, except for $\gamma(t)=3/\sqrt{2}$, the critical value of γ.

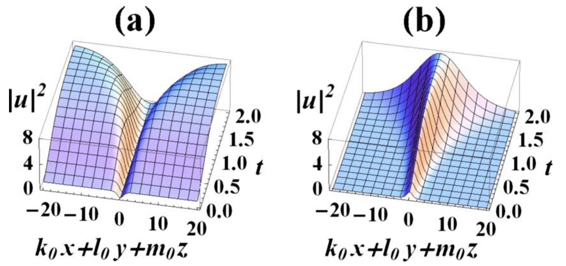

FIG. 5. (Color online) Straight soliton solutions as functions of time. The parameters are the same as in Fig. 4 except for $b_0=0$.

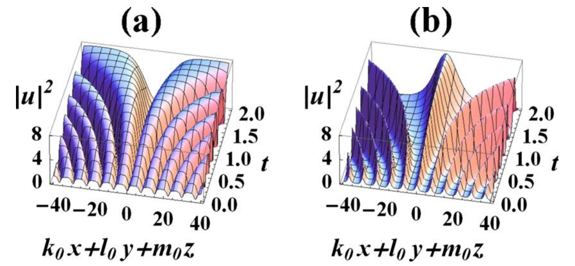

FIG. 6. (Color online) Traveling wave solutions in terms of JEFs. The parameters are the same as in Fig. 5, except for $M=0.99$. (a) $F=\text{sn}$ and (b) $F=\text{cn}$.

eter a. For $a_0>-\sqrt{\frac{\alpha}{2\beta}}$ the functions k, l, m, and b all converge to 0 and ω and e converge to constant values that depend on a_0.

We stress again the fact that the solutions obtained here in principle are transient (i.e., time-dependent) in nature. By inspecting Eq. (24) one can see that the long-time behavior of the general solution crucially depends on the coefficient $\gamma(t)$. Although $\gamma(t)$ is described as the linear gain or loss in the system, the value of $\gamma=0$ does not exert any special bearing on the solutions, similar to the value of $a_0=0$ for chirp. Figures 1–3 depict decaying solutions for a small negative value of γ. The solutions for $\gamma=0$ are also decaying.

The critical value of γ for the appearance of solitons or waves as t increases is $\gamma=3p/4$. Thus, if γ is constant, only for $\gamma=3p/4$ can one see stable solitons or waves evolving as $t\to\infty$. If $\gamma>3p/4$, the solutions blow up; if $\gamma<3p/4$, the solutions diminish. Hence, to observe solitons asymptotically, one needs *gain* in the system. To see periodically changing (breathing) solitons in the case of constant α and β, one needs γ in the form of $\gamma(t)=3p/4+\gamma_1(t)$, where $\gamma_1(t)$ is some periodic sign-changing real function.

The caveat to the analysis just presented is contained in Eq. (23), which explicitly connects $\chi(t)$ with $\gamma(t)$. Thus, the long-time behavior of the nonlinearity coefficient χ is also tied to $\gamma(t)$. For constant γ, the critical value now is $\gamma=p/4$. If $\gamma<p/4$, $\chi(t)$ diminishes with time, if $\gamma>p/4$, it blows up. Taken together with the result of the previous paragraph, it appears that the most interesting interval of γ for the long-time behavior is $p/4\leq\gamma(t)\leq 3p/4$; there the solutions decrease in time, while the nonlinearity coefficient increases. This statement reflects the difficulties in obtaining stable solitons in the *multidimensional* GP equation with constant coefficients. It is another reflection of the known difficulties with the wave stability and collapse in multidimensional NL Schrödinger equation [16]. Hence, to observe long-lived solitons, a delicate engineering in the form of $\gamma(t)$ is necessary.

The form of solutions in general depends on what JEFs are utilized. Table I lists some of JEFs (labeled from **1** to **10**) that may appear in the solutions. The parameter M varies between 0 and 1. When $M\to 0$, JEFs degenerate into trigonometric functions, and the periodic traveling wave solutions become the periodic trigonometric solutions. When $M\to 1$, JEFs degenerate into hyperbolic functions, and the traveling wave solutions become the ST soliton solutions. Figures 4–6 depict some of the typical examples.

We should note that for $M=1$ the solutions introduced by Eqs. (5)–(7) describe spatially *extended* ST solitons. Even though the amplitude A as a function of the *transverse* variable θ is localized, it is not when viewed in the plane of transverse coordinates x and y. This is easily seen if one rotates the x and y axes about the z axis for some angle α, to arrive at a set of new coordinates x' and y'. By choosing the angle as $\tan(\alpha)=-k/l$, the variable θ will not contain y', and by choosing $\tan(\alpha)=k/l$, it will not contain x'. Thus the amplitude A will not explicitly depend on y' (or x') and the soliton will be extended along the y' (or x') axis. Hence, the solutions obtained with the present method cannot be of the light bullet type [10].

V. CONCLUSION

In conclusion, we have solved analytically the (3+1)D GPE with distributed diffraction, nonlinearity, and gain. The case with constant diffraction and parabolic potential strength, but with variable gain, has been discussed in more detail. A number of exact traveling wave solutions have been found, and exact ST soliton solutions obtained. The influence of the chirp function on the phase and the amplitude of solutions is elucidated. The importance of the gain coefficient for the long-time behavior of solutions is emphasized. In particular, it is found that the positive gain is necessary for the existence of stable solitons when the coefficients are constant.

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